The Asymptotic Behavior of the Height for a Birth-Death Process

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Let $\{X_t, t \geq 0\}$ be the birth-and-death process with state space $E = \{0, 1, 2, \cdots, N\}$ and the following conservative Q-matrix: $Q = (q_{ij})$

$$q_{ij} = \begin{cases} -b_i, & j = i + 1, \quad 0 \leq i \leq N - 1, \\ -(a_i + b_i), & j = i, \quad 0 \leq i, \\ a_i, & j = i + 1, \quad 1 \leq i \leq N, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Where $a_0 = 0, a_i > 0, 1 \leq i \leq N, b_i > 0, 0 \leq i \leq N - 1, b_N = 0$.

Introduction

$\{t : X_t > 0\} = \cup_{i=1}^{\infty}[\tau_i, \eta_i)$, where $\{[\tau_i, \eta_i), i \in \mathbb{N}\}$ is the family of maximal disjoint random time intervals such that $X_t > 0$ on every interval $[\tau_i, \eta_i)$.

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$\{H_N^{(i)}, i \in \mathbb{N}\}$ i.i.d. $H_N^{(i)}$ is reduced to H_N.

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Suppose $b_i = (N-i)\nu, a_i = i\mu, \rho = \frac{\nu}{\mu}$, then

$$\lim_{N\to\infty} \frac{\mathbb{E}\left(H_N\right)}{N} = f(\rho). \tag{2}$$

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Where $\alpha := \alpha(\rho)$ be the unique solution of the equation $x^x(1-x)^{1-x} = \rho^x$ for $\rho \in (0,1)$.

$$f(\rho) = \begin{cases} \alpha, & 0 < \rho < 1, \\ 1, & \rho \geq 1. \end{cases} \tag{3}$$

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Theorem 1

Let $f(\rho)$ is given by (3), then

$$\lim_{N\to\infty} \frac{H_N}{N} = f(\rho) \text{ in } \mathrm{L}^2. \tag{4}$$

and

$$\lim_{N\to\infty} \frac{\mathrm{Var}\,(H_N)}{N} = \frac{f^2(\rho)}{\rho}, \tag{5}$$

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Theorem 2

Suppose $\varphi(x)$ satisfies that $\lim_{x\to\infty} \frac{\log x}{\varphi(x)} = 0$, then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{H_N - \mathbb{E}(H_N)}{\varphi(N)} \leq x\right) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{6}$$

The main results

Theorem 3

Suppose $b_i = \nu(N-i)^\beta, a_i = \mu i^\beta, \rho = \frac{\nu}{\mu}$, then

$$\lim_{N\to\infty} \frac{H_N}{N} = f(\rho, \beta) \ \text{ in } \mathrm{L}^2. \tag{7}$$

Further, when $\beta > 1$, so that

$$\lim_{N\to\infty} \frac{H_N}{N} = f(\rho, \beta) \ a.s. \tag{8}$$

Where $\alpha := \alpha(\rho, \beta)$ is the unique solution of the equation $x^x(1-x)^{1-x} = \rho^{x/\beta}$ for $\rho \in (0, 1)$.

$$f(\rho, \beta) = \begin{cases} \alpha, & 0 < \rho < 1, \\ 1, & \rho \geq 1. \end{cases} \tag{9}$$

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Theorem 4

(1) If $0 < \beta < 2$, then

$$\lim_{N\to\infty} \frac{\mathrm{Var}\,(H_N)}{N^{2-\beta}} = \frac{f^2(\rho,\beta)}{\rho}. \tag{10}$$

(2) If $\beta \geq 2$, then

$$\mathrm{Var}\,(H_N) \leq C \log^2 N. \tag{11}$$

Where C is independent of N, depend on β, ρ.

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Theorem 5

The finite queue $M/M/s/N$, $a_i = i\mu, i = 1, 2, \cdots, s, a_i = s\mu$, $i > s$, $b_i = \lambda$. Let $\rho = \lambda/(s\mu)$,

(1) If $0 < \rho < 1$, then

$$\lim_{N\to\infty} \mathbb{E}(H_N) = \psi(\rho), \quad \lim_{N\to\infty} \mathrm{Var}(H_N) = \varphi(\rho). \tag{12}$$

(2) If $\rho = 1$, then

$$\lim_{N\to\infty} \frac{\mathbb{E}(H_N)}{\log N} = \frac{s^{(s-1)}}{(s-1)!}, \quad \lim_{N\to\infty} \frac{\mathrm{Var}(H_N)}{N} = \frac{2s^{(s-1)}}{(s-1)!}. \tag{13}$$

(3) If $\rho > 1$, then

$$\lim_{N\to\infty} \frac{\mathbb{E}(H_N)}{N} = \zeta(\rho), \lim_{N\to\infty} \frac{\mathrm{Var}(H_N)}{N^2} = \eta(\rho). \tag{14}$$

Where $\psi(\rho), \varphi(\rho), \zeta(\rho), \eta(\rho)$ is finite, depend on s, ρ.

Distribution of H_N

Lemma 1

The birth-death process X_t with Q-matrix (1), distribution of H_N for X_t follows:

$$\mathbb{P}(H_N \geq k) = \frac{1}{1 + \sum_{i=1}^{k-1} \frac{a_i \cdots a_1}{b_i \cdots b_1}}, k = 2, 3, \cdots, N. \tag{15}$$

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Let $a_i = i^\beta \mu$, $b_i = (N-i)^\beta \nu$, then

$$\mathbb{P}(H_N \geq k) = \frac{1}{1 + \sum_{i=1}^{k-1} \rho^{-i} \binom{N-1}{i}^{-\beta}}, k = 2, 3, \cdots, N. \tag{16}$$

$$r_{\rho,\beta,n}(i) = \rho^{-i} \binom{n-1}{i}^{-\beta}$$

Lemma 2

Let $\alpha := \alpha(\rho, \beta)$ be the unique solution of the equation $x^x(1-x)^{1-x} = \rho^{x/\beta}$ for $\rho \in (0,1)$, $h_n = [\alpha(n-1)]$, then

$$r_{\rho,\beta,n}(h_n) = O\left(n^{\beta/2}\right). \tag{17}$$

Proof of Lemma 6

Lemma 3

Suppose $\rho \in (0,1)$, there exists some constants C_1, C_2, such that

$$r_{\rho,\beta,n}\left(h_n + [C_1 \log n]\right) \geq r_{\rho,n}(h_n)n^2, \tag{18}$$

$$r_{\rho,\beta,n}\left(h_n - [C_2 \log n]\right) \leq r_{\rho,n}(h_n)n^{-3} \tag{19}$$

for large enough n.

Estimate of $\mathbb{E}(H_N)$

Lemma 4

(1) If $0 < \rho < 1$, there exists some constants C_3, C_4, for N large enough, we have

$$[\alpha N] - C_3 N^{1-\beta} - C_4 \log N \leq \mathbb{E}(H_N) \leq [\alpha N] + 1. \tag{20}$$

(2) If $\rho \geq 1$, there exists some constants C_5, for N large enough, so that

$$N - C_5 N^{1-\beta} \leq \mathbb{E}(H_N) \leq N. \tag{21}$$

Estimate of $\mathrm{Var}(H_N)$

Lemma 5

(1) If $0 < \rho < 1$, for N large enough, we have

$$\frac{([\alpha N] - [C_2 \log N] - C_3 N^{1-\beta})^2}{1 + \rho(N-1)^\beta} \leq \mathrm{Var}(H_N)$$

$$\leq \frac{\alpha^2}{\rho} N^{2-\beta} + O\left(N^{2-2\beta}\right) + C \log^2 N.$$

(2) If $\rho \geq 1$, for N large enough, we have

$$\frac{(N - 2\rho^{-1}(N-1)^{1-\beta} - 1)^2}{1 + \rho(N-1)^\beta} \leq \mathrm{Var}(H_N)$$

$$\leq \rho^{-1} N^2 (N-1)^{-\beta} + O\left(N^{2-2\beta}\right) + C \log^2 N.$$

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(1) If $0 < \rho < 1$, for N large enough, we have

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$$\leq \rho^{-1} N^2 (N-1)^{-\beta} + O\left(N^{2-2\beta}\right) + C\log^2 N.$$

By lemma 5, we proof Theorem 4.

Proof of Theorem 3

$$\mathbb{E}\left|\frac{H_N}{N} - f(\rho)\right|^2 = \mathrm{Var}\left(\frac{H_N}{N}\right) + \left|\frac{\mathbb{E}(H_N)}{N} - f(\rho)\right|^2 = O\left(N^{-\beta}\right).$$

$$\lim_{N\to\infty} \frac{H_N}{N} = f(\rho) \quad \text{in } L^2.$$

When $\beta > 1$, $\forall \varepsilon > 0, \mathbb{P}\left(\left|\frac{H_N}{N} - f(\rho)\right| > \varepsilon\right) = O\left(\frac{1}{N^\beta \varepsilon^2}\right)$, then

$$\sum_{N=1}^{+\infty} \mathbb{P}\left(\left|\frac{H_N}{N} - f(\rho)\right| > \varepsilon\right) < +\infty.$$

by Borel-Cantelli lemma, then

$$\lim_{N\to\infty} \frac{H_N}{N} = f(\rho) \quad a.s.$$

Proof of Theorem 5

The finite queue $M/M/s/N$, $a_i = i\mu, i = 1, 2, \cdots, s, a_i = s\mu$, $i > s$, $b_i = \lambda$. Let $\rho = \lambda/(s\mu)$, 令$r_0 = 1, r_i = \dfrac{a_i a_{i-1} \cdots a_1}{b_i b_{i-1} \cdots b_1}$, then

$$r_i = \begin{cases} \dfrac{i!}{s^i} \rho^{-i}, & i < s, \\ \dfrac{(s-1)!}{s^{s-1}} \rho^{-i}, & i \geq s. \end{cases} \tag{22}$$

by lemma 1, we have

$$\mathbb{P}\left(H_N \geq k\right) = \frac{1}{\sum_{i=0}^{k-1} r_i}. \tag{23}$$

$$\mathbb{E}\left(H_N\right) = \sum_{k=1}^{N} \frac{1}{\sum_{i=0}^{k-1} r_i}. \tag{24}$$

$\forall k \in \{1, 2, \cdots, N-1\}$, we have

$$\mathbb{P}(H_N = k) = \mathbb{P}(H_N \geq k) - \mathbb{P}(H_N \geq k+1) \tag{25}$$

$$= \frac{r_k}{\sum_{i=0}^{k-1} r_i \cdot \sum_{j=0}^{k} r_j}. \tag{26}$$

$$\begin{aligned}\mathbb{E}\left(H_N^2\right) = \sum_{k=1}^{N} k^2 \cdot \mathbb{P}(H_N = k) = \sum_{k=1}^{s} k^2 \cdot \frac{r_k}{\sum_{i=0}^{k-1} r_i \cdot \sum_{j=0}^{k} r_j} \\ + \sum_{k=s+1}^{N-1} k^2 \cdot \frac{r_s}{\left(\sum_{i=0}^{s-1} r_i + (k-s) r_s\right)\left(\sum_{i=0}^{s-1} r_i + (k-s+1) r_s\right)} \\ + N^2 \cdot \frac{1}{\sum_{i=0}^{s-1} r_i + (N-s) r_s}.\end{aligned} \tag{27}$$

References

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THANK YOU!